Topology Vol. 24, No. 3, pp. 333–340, 1985.
Printed in Great Britain.

0040–9383/85 $3.00+.00

VECTOR FIELDS WITH TRANSVERSE FOLIATIONS†

SUE GOODMAN

(Received 6 February 1984)

THERE are many examples of nonsingular vector fields which are not transverse to any codimension one foliation. The Hopf fibration of S^3 is one example. That it has no transverse foliation is shown as follows. By Novikov [11], any foliation of S^3 has a Reeb component. Any flow transverse to a Reeb component has nonperiodic orbits, since any orbit meeting the boundary torus can never again leave the Reeb component to close up. (Also see [4] for a more elementary proof.)

It has been unknown however whether the set of nonsingular vector fields which have an everywhere transverse foliation is dense in the C^1—topology of all nonsingular vector fields on a manifold. The Hopf flow, for example, has flows arbitrarily near it with transverse foliations. Thurston constructed an example of a particular 3-dimensional manifold where this set of vector fields is not dense. In §1 and §2 of this paper, it will be shown that in any homotopy class of nonsingular vector fields on every closed oriented manifold, there is an open set of vector fields which are not transverse to any foliation of codimension one. The results in §1 and 2 are joint work with D. Asimov [2a]. Since the work appears only in preprint form, it is included here. The generalization to higher dimensions was done jointly with Paul Schweitzer.

The nonexistence of a transverse foliation in dimension 3 follows from the failure of a certain linking property between periodic orbits of the flow. The necessity of this linking property is shown in §1. In §2, we construct nonsingular flows which do not have the linking property nor does any flow near them.

However the linking property is not in general sufficient for a flow to have a transverse foliation. For example the Hopf flow on S^3 satisfies the linking property. In §3 it is shown however that for nonsingular Morse–Smale flows it is sufficient. The foliations, in some cases, are not of class C^2 and in §4, it is shown that this lack of smoothness cannot be avoided.

The joint work with D. Asimov and P. Schweitzer was announced in [2b] and the sufficiency of the linking property for Morse–Smale flows in [6].

§1. A NECESSARY CONDITION FOR A FLOW TO HAVE A TRANSVERSE FOLIATION IN DIMENSION 3

Let Φ be a nonsingular flow on a closed oriented 3-manifold M. We will show that in order for Φ to be transverse to a 2-dimensional foliation of M, Φ must satisfy the following property:

Linking Property. If there is a periodic orbit σ of Φ which bounds an imbedded 2-disk D in M then the interior of D must intersect a periodic orbit.

We will use the following result of Novikov [11];

LEMMA 1.1. *Any nonsingular flow which is transverse to a Reeb component R (on $D^2 \times S^1$) has a periodic orbit contained in interior R and homotopic to the generator of $\pi_1(D^2 \times S^1)$.*

THEOREM 1.2. *Given a nonsingular flow Φ with a foliation F everywhere transverse to Φ, then Φ has the linking property.*

†Partially supported by NSF Grant #MCS–8100785.

Proof. Suppose Φ does not have the linking property, i.e. suppose there is a periodic orbit σ bounding an imbedded disk D in M and interior D meets no periodic orbit. Further suppose there is a foliation F transverse to Φ.

Since interior D meets no periodic orbit, by Lemma 1.1, D cannot contain any meridian circle of a Reeb component. A sufficiently small perturbation of D, keeping $\partial D = \sigma$ fixed, will also have this property. Choose one, D^1, in general position with respect to F (see [5]). We have then a contradiction with a result of Novikov [11] which says that if a closed transverse curve to a foliation (σ in this case) bounds an imbedded disk in general position with respect to the foliation, then that disk contains a vanishing cycle and hence a meridian circle of a Reeb component.

§2. CONSTRUCTION OF STABLE NONSINGULAR FLOWS NOT TRANSVERSE TO ANY FOLIATION

In this section, we construct an open set of flows which do not have the linking property, hence do not have transverse foliations. In particular, there exist nonsingular Morse–Smale flows with no transverse foliation.

THEOREM 2.1. *In any homotopy class of nonsingular vector fields on any closed oriented 3-manifold, there is an open set of vector fields without the linking property.*

Proof. Let Φ be a nonsingular C^1 flow on M^3. Since M is compact, a small homotopy of Φ produces a flow Φ' with a periodic orbit γ. Take a sufficiently small solid toral neighborhood of γ so that Φ' is transverse to the disk fibers of the neighborhood. In this neighborhood Φ' can easily be homotoped (to Φ'') to make γ an attracting periodic orbit. A smaller solid toral neighborhood T of γ can be chosen so that ∂T is transverse to Φ'' and every orbit meeting T limits on γ.

One further modification is needed. Let $A \cong S^1 x [-1, 1] x [-1, 1]$ be a thickened annulus contained in a flow box for Φ'' so that $A \cap T = S^1 x [-1, 1] x \{-1\}$, where $S^1 x \{0\} x \{-1\}$ bounds a disk D on ∂T. In the interior of A, homotope Φ'' to a nonsingular Morse–Smale flow ψ such that there are precisely two periodic orbits in A, an attractor and a saddle, each homotopic to the S^1-factor in A. See Fig. 1. Let σ be the saddle, with coordinates $S^1 x \{0\} x \{-\frac{1}{4}\}$.

Since A is contained in a flow box, σ is null-homotopic in M. In fact it bounds a disk D' given by $(S^1 x \{0\} x [-1, -\frac{1}{4}]) \cup D$ (the lower half of $\{W^u(\sigma) \cap A\}$ union D). Note that interior D' does not meet a periodic orbit, since every orbit meeting interior D' meets T and hence has the sink γ as its ω-limit set. If follows that ψ, homotopic to Φ, does not have the linking property (Fig. 2).

Finally, since $\psi | A \cup T$ is Morse–Smale, this property is stable under small perturbations. Therefore an open neighborhood of ψ has this property.

COROLLARY 2.2. *On any closed oriented 3-manifold, in any homotopy class of nonsingular vector fields, there is an open set with no transverse foliations.*

Note. Orientability of the manifold is probably an unnecessary assumption. One must allow for the possibility of an "unorientable Reeb component" of a twisted disk bundle over

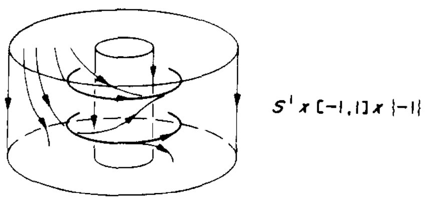

Fig. 1.

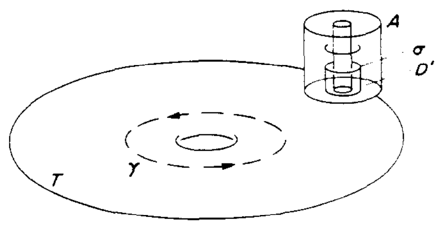

Fig. 2.

S^1. Analogues of lemma 1.1 and of Novikov's theorem should still hold and the rest follows as above.

Note. Paul Schweitzer and the author have shown that a similar construction can be done in higher dimensions to create open sets of vector fields in any homotopy class which have no transverse foliations. Roughly the idea is as follows:

Again homotope the original vector field to create an attracting periodic orbit in a neighborhood $D^{n-1} \times S^1$ (in an n-dimensional manifold, $n > 3$) with the flow transverse to the boundary and all orbits limiting on the sink. Take $S^2 \times [-1, 1] \times S^1 \subseteq D^{n-1} \times S^1$ with each $(S^2 \times \{\text{point}\} \times S^1)$ transverse to the flow. Now homotope so that $S^2 \times \{0\} \times S^1$ is tangent to the flow and has a nonsingular Morse–Smale flow with a single attracting periodic orbit and a single repelling periodic orbit induced on it. Further we want the flow near $(S^2 \times \{0\} \times S^1)$ to point inward toward $(S^2 \times \{0\} \times S^1)$ (i.e., there is a neighborhood, $(S^2 \times (-\varepsilon, \varepsilon) \times S^1$ with boundary transverse to the flow and every orbit limiting on $S^2 \times \{0\} \times S^1$). Now homotope one final time to create in $S^2 \times \{0\} \times S^1$ the annulus A near the attractor as in Theorem 2.1. If this final flow were to have a transverse foliation, there would be a transverse foliation induced on $S^2 \times \{0\} \times S^1$, a contradiction. The flow on $S^2 \times (-\varepsilon, \varepsilon) \times S^1$ is stable under small perturbations, yielding an open set of flows on M^n with no transverse foliation. Therefore we have

COROLLARY 2.3. *In any homotopy class of nonsingular vector fields of any closed oriented n-manifold, there is an open set with no transverse foliations.*

Note that the linking property does not directly apply to the flow on M^n since Novikov's theorem is 3-dimensional.

§3. SUFFICIENCY OF THE LINKING PROPERTY FOR NONSINGULAR MORSE–SMALE FLOWS

The main theorem of this section is the following:

THEOREM 3.1. *Let Φ be a nonsingular Morse–Smale flow on a closed oriented 3-manifold M. Then Φ has a foliation everywhere transverse to it if and only if it satisfies the linking property.*

Necessity of the linking property follows from §1. In this section we prove sufficiency by constructing the transverse foliation. Associated with Φ is a Lyapunov function (see [3], [12]) i.e. a smooth function $f: M \to \mathbb{R}$ such that (i) $\frac{d}{dt}(f(\Phi_t(x))) < 0$ if x is not in the set of periodic orbits of Φ and (ii) $f(x) = f(y)$ if and only if x, y belong to the same periodic orbit of Φ. Let $C_0 < \ldots < C_r$ be the critical levels of f corresponding to the periodic orbits $\sigma_0, \ldots, \sigma_r$ of the flow Φ, each of which is an attractor, repeller or saddle. The Lyapunov function gives rise to a filtration, a collection of submanifolds $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ such that

(a) $\Phi(M_i) \subseteq$ interior M_i and

(b) $\sigma_i = \bigcap_t \Phi_t(M_i - M_{i-1})$.

The boundaries of the M_i's are simply $f^{-1}(r_i)$, the level surfaces corresponding to regular values r_i separating the critical values $(C_0 < r_0 < C_1 < \ldots < r_{r-1} < C_r)$.

We begin the construction of the foliation with the attracting closed orbits, say $\sigma_0, \ldots, \sigma_k$. The submanifolds M_0, $M_1 - M_0, \ldots,$ $M_k - M_{k-1}$ are solid tori, each homeomorphic to $D^2 \times S^1$, with the flow oriented in on each boundary torus, and every orbit entering a component has the attractor σ_i as its ω-limit set. We foliate each solid torus with a Reeb component transverse to the flow, with the boundary of the Reeb component being ∂M_i, as illustrated in Fig. 3.

We now proceed to the saddles, $\sigma_{k+1}, \ldots, \sigma_l$. If σ_j is an untwisted saddle, $M_j - \text{int}\, M_{j-1}$ contains a neighborhood homeomorphic to $[-1, 1] \times [-1, 1] \times S^1$, with $\sigma_j \cong \{0\} \times \{0\} \times S^1$ and the flow transverse inward on $\{-1, 1\} \times (-1, 1) \times S^1$ and transverse outward on $(-1, 1) \times \{-1, 1\} \times S^1$ (a round 1-handle in the sense of Asimov [1]). We may assume that $[-\frac{3}{4}, \frac{3}{4}]$

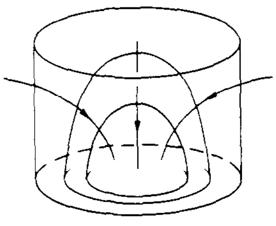

Fig. 3.

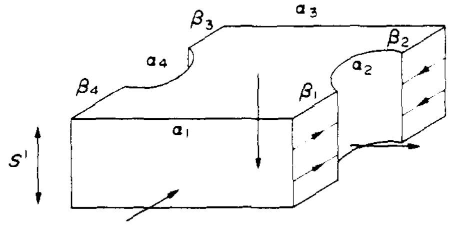

Fig. 4.

$x\{-1, 1\} x S^1$ is contained in ∂M_{j-1} and $\{-1, 1\} x [-\frac{3}{4}, \frac{3}{4}] x S^1$ and ∂M_j and that the orbits entering $\{-1, 1\} x \{-\frac{3}{4}, \frac{3}{4}\} x S^1$ exit through $\{-\frac{3}{4}, \frac{3}{4}\} x \{-1, 1\} x S^1$. In fact we may further assume that flow is horizontal here, i.e. an orbit entering at say $\{-1\} x \{-\frac{3}{4}\} x \{\theta\}$ exits at the same level θ of S^1. Hence contained in the round 1-handle is a neighborhood of σ_j homeomorphic to an octagon $x S^1$, where the edges of the octagon $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_4, \beta_4$ are such that the flow is transverse inward on $\alpha_1 x S^1 \cup \alpha_3 x S^1 \subseteq \partial M_j$, transverse outward on $\alpha_2 x S^1 \cup \alpha_3 x S^1 \subseteq \partial M_{j-1}$ and tangent to $\beta_i x S^1$ (with orbits $\beta_i x \{\theta\}$). See Fig. 4. Call this neighborhood K.

In the case of a twisted saddle, where the local stable and unstable manifolds are Möbius strips, there is a neighborhood similar to this, but obtained by taking an octagon $x[-1, 1]$ and identifying top to bottom with a 180° twist. So we have a twisted octagon bundle over S^1, with the edges alternating as above. This neighborhood will also be denoted by K.

In the first (untwisted) case, ∂K consists of eight annuli, four tangent to the flow, two contained in ∂M_j and two in ∂M_{j-1}. In the twisted case there are only four annuli in ∂K, two tangent to Φ, one contained in ∂M_j and one in ∂M_{j-1}. In either case, K can readily be foliated transverse to the flow and, for now, transverse to ∂K. The foliation will in fact be a fibration on K with leaves homeomorphic to the octagonal disk fibers, but bent up toward the boundary annuli where the flow is transverse outward and down where it is transverse inward, to produce saddle-shaped leaves as shown in Fig. 5. It remains to foliate $(M_j - \operatorname{int} M_{j-1}) - K$. We shall construct a foliation transverse to the boundary of $M_j - \operatorname{int} M_{j-1}$ and then turbularise to make the boundary tangent to the foliation.

We begin with the first saddle $\sigma_{k+1} \subseteq M_{k+1} - M_k$. Each of the annuli $\alpha_2 x S^1$ and $\alpha_4 x S^1$ (or a single annulus if σ_{k+1} is a twisted saddle) is part of ∂M_k, which consists of tori (the boundaries of the Reeb components surrounding the attractors). Further because we are assuming that the flow Φ has the linking property, these annuli must be homotopically essential in ∂M_k. Otherwise σ_{k+1}, freely homotopic to the core circle of the annuli, would be null-homotopic in $M_{k+1} - M_k$ and could not link a periodic orbit.

So for the untwisted saddle, the two annuli where the flow is transverse outward are glued to parallel essential annuli on the same toral component T_1 of ∂M_k or to essential annuli on two different components T_1, T_2 of ∂M_k. In the twisted case, the one annulus is glued to an essential annulus of some component T_1 of ∂M_k. In either case, $\partial M_k - \partial K$ consists of annuli $\subseteq T_1, T_2$ plus possibly some tori (not meeting K).

The flow on $(M_{k+1} - \operatorname{int} M_k) - K$ is very simple: orbits enter through ∂M_{k+1} and exit through ∂M_k. Hence there is a homeomorphism (given by following the orbits) of ∂M_{k+1}

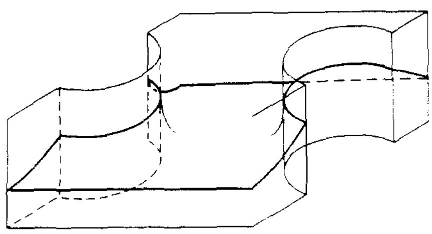

Fig. 5.

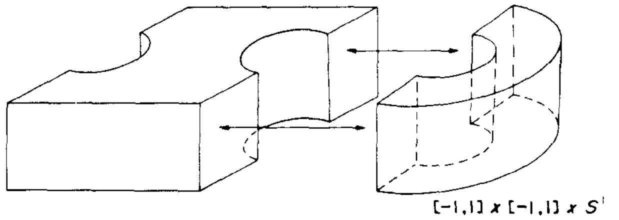

Fig. 6.

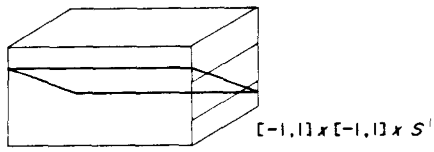

Fig. 7.

$-\partial K$ with $T_1 - \partial K$ or $T_1 \cup T_2 - \partial K$ (which as we have noted consists of one or two annuli). Hence $(M_{k+1} - \text{int}\, M_k) - K$ is a union of one or two pieces each homeomorphic to $[-1, 1] \times [-1, 1] \times S^1$ where $\{-1, 1\} \times [-1, 1] \times S^1$ is tangent to the flow, identified with two of the annuli in K tangent to the flow, and $[-1, 1] \times \{-1, 1\} \times S^1$ is transverse to the flow. In fact the orbits of the flow restricted to each of these connecting pieces may be assumed to be $\{x\} \times [-1, 1]$ where $x \in \partial M_{k+1} - \partial K$.

We now want to foliate these connecting pieces, transverse to the boundary, so that the foliation on $\{-1, 1\} \times [-1, 1] \times S^1$ matches the foliation on ∂K where it is to be identified. There are two cases. The first, and simplest, is where the connecting piece $[-1, 1] \times [-1, 1] \times S^1$ is glued to K without a twist. (See Fig. 6). For a twisted saddle, this is the only choice. The foliation on the connecting $[-1, 1] \times [-1, 1] \times S^1$ is simply a product as shown in Fig. 7. The submanifold $M_{k+1} - \text{int}\, M_k$ is a fiber bundle over S^1 with fiber a disk minus two smaller disks. The bundle is twisted or not according to whether the saddle is twisted or not. The induced foliation on $\partial M_{k+1} \cup \partial M_k$ is a trivial foliation by circles. It can easily be turbularized so that $\partial M_{k+1} \cup \partial M_k$ is tangent to the foliation, consisting of toral leaves. A typical leaf at the foliation, after turbularization is sketched below (Fig. 8).

The second possibility is that the connecting piece, $[-1, 1] \times [-1, 1] \times S^1$ is glued to an untwisted saddle with a twist, i.e. to $\beta_1 \times S^1 \cup \beta_3 \times S^1$ or $\beta_2 \times S^1 \cup \beta_4 \times S^1$. There will then be another twisted connector joining the remaining pair on ∂K. In this case the foliation induced on $\{1\} \times [-1, 1] \times S^1$ by ∂K has the opposite slope of the foliation induced on $\{1\} \times [-1, 1] \times S^1$. We will alter the original foliation on K so that the slopes of the foliation induced on $\{-1, 1\} \times [-1, 1] \times S^1$ by ∂K are the same. Then the connecting piece can be foliated as in the first, untwisted case, to complete the foliation of $M_{k+1} - M_k$.

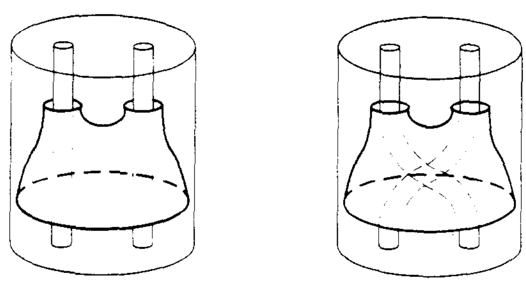

Fig. 8.

The alteration of the foliation of K will consist of erecting a wall (which will be a leaf) to allow for the change of slope. Consider the unstable manifold of the saddle σ_{k+1}, $W^u(\sigma_{k+1})$, intersected with K. It is an annulus in this case, with the flow tangent to it. Erect the wall parallel to σ_{k+1} but to one side of it, and transverse to the existing foliation on K, stretching from one component of $\partial M_k \cap K$ to the other, as shown in Fig. 9.

Cut K along this wall and invert one side. The orientation of the foliations on either side of the wall will not match. So near the wall we sweep the leaves up to approach the wall asymptotically, making the wall a leaf and allowing us to reglue along the wall. This new foliation on K will induce foliations on $\{-1, 1\} x [-1, 1] x S^1$ of the connecting piece which have the same slope. See Fig. 9.

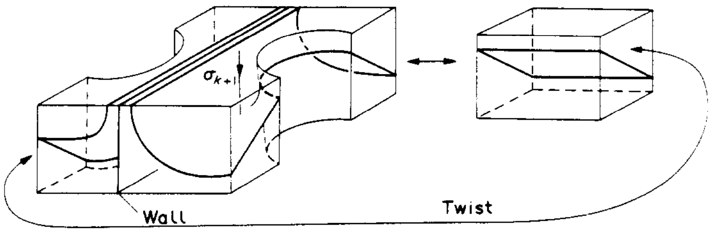

Fig. 9.

Notice that in this case the component $M_{k+1} - M_k$ is a twisted S^1-bundle (the unit tangent bundle) over a Möbius band minus a disk.

The foliations induced on the boundary tori have two compact circle leaves (one from each wall in the connecting pieces), with the other leaves approaching these circles asymptotically, as shown in Fig. 10. So the foliation may again be spun near the boundary tori to approach the boundary asymptotically, once again giving a foliation of $M_{k+1} - \text{int}\, M_k$ transverse to the flow and tangent to the boundary. (See [8] for a special case of this construction.)

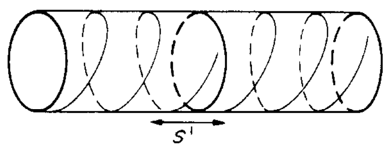

Fig. 10.

In either case, we now have the desired foliation of M_{k+1}. We repeat this process on each $M_j - M_{j-1}$ $(k+1 \leq j \leq l)$, noting that ∂M_j is always a union of toral leaves. At the $j = l$ stage, we have a foliation of M_l, whose closed leaves are tori, where each torus is a component of a level set $f^{-1}(r_i)$. Further ∂M_l is a union of toral leaves. What remains is the set of repellers $\{\sigma_{l+1}, \ldots, \sigma_r\}$. Each of these has a neighborhood homeomorphic to $D^2 \times S^1$ and can be foliated by a Reeb component transverse to Φ. The boundaries of the Reeb components are the components of ∂M_l. This completes the foliation of M, transverse to Φ.

§4. SMOOTHNESS OF THE FOLIATION

In the construction of section 3, if each connecting piece is glued to ∂K without a twist, the foliation is clearly C^∞. However this is not true if there is a twist. Then the foliation is only C^1. This can be seen as follows.

Recall that when there is a twist in a connecting piece, the foliation induced on the boundary torus T, before turbularization, has two compact leaves, as shown in Fig. 10. When this foliation is spun around T to approach it asymptotically, each compact leaf produces a cylindrical end, homeomorphic to $(0, \infty) \times S^1$. The other leaves near it will have planar ends, homeomorphic to $(0, \infty) \times \mathbb{R}$, spiralling around the cylindrical end as they approach T. Now consider the one-sided holonomy of the leaf T. One generator of the holonomy, taken in the direction of spinning, is clearly a contraction. The other generator of holonomy can be taken to be homotopic to the S^1-factor in the cylindrical leaf end. In that direction, after turbularization, the holonomy map has a sequence of fixed points approaching the leaf T, with contractions between the fixed points.

Since the fundamental group of the torus T is abelian, we have two commuting diffeomorphisms, of $[0, 1]$ (the holonomy maps described above) both fixing 0, one of which is a contraction and the other having a sequence of fixed points approaching 0. By a theorem of Kopell [7], if the maps are C^2, the second must be the identity, which it is not. So the foliation is not C^2.

The following question is then raised: is there a C^2 foliation on the neighborhood of the saddle, homeomorphic to the unit tangent bundle of a Möbius band minus a disk, which is transverse to the flow and tangent to the boundary? The answer is no.

THEOREM 4.1. *If $M_j - M_{j-1}$ (in the notation of §3), a submanifold of a filtration containing a saddle σ_j of a Morse–Smale flow, is homeomorphic to the unit tangent bundle of a Möbius band minus a disk, there is no C^2 foliation of $M_j - M_{j-1}$ which is tangent to the boundary and transverse to the flow.*

Proof. Suppose F is a foliation of $M_j - M_{j-1}$, transverse to the flow and tangent to the boundary. The boundary of $M_j - M_{j-1}$ consists of two toral leaves. It may be assumed that these leaves are the only compact leaves. This is because any other compact leaf must cobound $T^2 \times [0, 1]$ with one of the boundary leaves, hence we could restrict to that smaller component.

The stable and unstable manifolds of σ_j divide $M_j - M_{j-1}$ into two connected components which we will denote I and II. Each piece is homeomorphic to $[0, 1] x [0, 1] x S^1$ with $\{0, 1\} x \{0, 1\} x S^1$ tangent to the foliation (consisting of half of ∂M_j and half of ∂M_{j-1}) and $[0, 1] x \{0, 1\} x S^1$ tangent with each end consisting of half the unstable manifold of σ_j and half the stable. These pieces are shown in Fig. 11, with the letters indicating identifications of halves of the stable or unstable manifold. Except for the edges, every orbit of the flow enters the top and flows downward, existing through the bottom. The foliation on each piece must be topologically conjugate to the product with $[0, 1]$ of the foliation induced on $C \cup D$ (respectively $A \cup D$ on piece II). Hence every leaf of I or II is homeomorphic to $S^1 x [0, 1]$ or $\mathbb{R} x [0, 1]$, depending upon whether that leaf intersects $C \cup D$ (or $A \cup D$) in a circle or a line (see [9], [10]).

Any leaf meeting σ_j on $F|C \cup D$ is homeomorphic to $\mathbb{R}$, hence spirals toward two circle leaves of $F|C \cup D$ freely homotopic to the boundary of the annulus $C \cup D$. Note however that these two circle leaves cannot be $\partial(C \cup D)$ since σ_j has opposite orientations on the two sides of I, and the foliation would not be transverse to the flow. So at least one of the circle leaves cannot be on the boundary of $C \cup D$. Let us assume that there is a circle leaf in the interior of C, call it τ_1. Further denote by L_1 the leaf of $F|I$ that it is on.

L_1 is homeomorphic to $S_1 x[0, 1]$, so $\partial L_1 = \tau_1 \cup \tau_2$, $\tau_2 \cong S^1$, and $\tau_2 \subseteq A \cup B$. Suppose $\tau_2 \subseteq A$. Then τ_2 corresponds to a circle leaf of $F|_{A \cup D}$ in piece II. And $\tau_2 \subseteq \partial L_2$, where $L_2 = S^1 x [0, 1]$ is a leaf of $F|\text{II}$ and $\partial L_2 = \tau_2 \cup \tau_3$, $\tau_3 \subseteq C \cup B \subseteq \text{II}$. Now if $\tau_3 \subseteq C$, $\tau_3 \neq \tau_1$

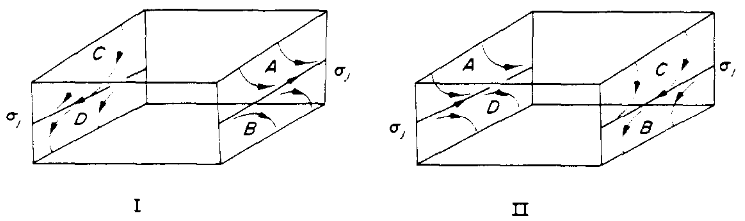

Fig. 11.

since there are no compact leaves in the interior of $M_j - M_{j-1}$; so τ_3 lies between τ_1 and $\partial(C \cup D)$ in I. Continuing this process, we produce a cylindrical end of a leaf $L_1 \cup L_2 \cup \ldots$ of F which eventually limits on ∂M_j.

If in the above argument, $\tau_3 \subseteq B$ in piece II, instead of C, we would produce a cylindrical end of a leaf of F limiting on ∂M_{j-1}. Also one could argue similarly that if τ_2 were contained in B instead of A at the first step, we would reach the same conclusion: a cylindrical end ($\cong S^1 \times [0, \infty]$) approaching either ∂M_j or ∂M_{j-1}. Also notice that this cylindrical end has leaves with planar ends ($\cong \mathbb{R} \times (0, \infty)$) spiralling around it while approaching ∂M_j or ∂M_{j-1}.

In any case, we see the same behavior near a boundary toral leaf as we saw at the beginning of this section; that is, contracting holonomy in one generating direction and a sequence of fixed points in the holonomy map in another generating direction, but not the identity in this second direction. Hence the foliation cannot be C^2.

We have from this the following easy corollary.

Corollary 4.2. *There exist closed oriented 3-manifolds with nonsingular Morse–Smale flows which have a C^1 transverse foliation but no C^2 transverse foliation.*

Proof. Let N be the unit tangent bundle of the Möbius band minus a disk with a flow as described in Theorem 4.1. Complete to a nonsingular Morse–Smale flow on a closed oriented 3-manifold by gluing a 0-handle to one boundary component and a 2-handle to the other. This flow will be Morse–Smale and will satisfy the linking property, hence will have a transverse foliation by the Theorem 3.1. The attractor and repeller (in the 0-handle and 2-handle respectively) are transverse to the foliation, so we may remove a solid toral neighborhood of each, whose boundary is transverse to the foliation, (including a trivial foliation on the boundary torus). We can then turbularize near the boundary of the remaining piece to produce a foliation of a manifold homeomorphic to $N(= M_1 - M_0$ for the flow), tangent to the boundary of N and transverse to the flow. By Theorem 4.1, it cannot be C^2, hence the original foliation was not.

Remark. Via corollary 4.2, we produce an example of a C^1-smooth foliation not approximable by C^2-smooth foliations.

BIBLIOGRAPHY

1. D. Asimov: Round handles and nonsingular Morse–Smale flows, *Ann. of Math.* **102** (1975), 41–54.
2. (a) D. Asimov and S. Goodman: Stable nonsingular flows not transverse to any foliation, (preprint 1978); (b) D. Asimov, S. Goodman and P. Schweitzer: Stable nonsingular flows not transverse to any foliation, *Abs. Am. Math. Soc.* **1** (1980), 139.
3. C. Conley: Isolated invariant sets and the Morse index, *C.B.M. A. Regional Conf. Series in Math.* **38**, A.M.S. (1978).
4. C. Ehresmann: Sur les éspaces fibrés différentiables, *C. R. Acad. Sci. Paris* **224** (1947), 1611–12.
5. J. Franks: Anosov diffeomorphisms, *Proc. Symp. Pure Math.* **14** (1980), 61–93.
6. S. Goodman: Vector fields with transverse foliations, *Abs. Am. Math. Soc.* **2** (1981), 592.
7. N. Kopell: Commuting diffeomorphisms, *Proc. Symp. Pure Math.* **14** (1970), 165–184.
8. B. Lawson: Foliations, *Bull. A.M.S.* **80** (1974), 369–418.
9. R. Moussu: Sur les feuilletages de codimension un, Thesis, Univ. of Paris (1971).
10. R. Moussu and F. Pelletier: Sur le theoreme de Poincare—Bendixson, *Anna. Inst. Fourier* **24** (1974), 131–148.
11. S. P. Novikov: Topology of foliations, *Trans. Moscow Math. Soc.* **14** (1965), 248–278; AMS Translation (1976), pp. 268–304.
12. W. Wilson: Smoothing derivatives of functions and applications, *Trans. Am. Math. Soc.* **139** (1969), 413–428.

Department of Mathematics,
University of North Carolina,
Chapel Hill, NC 27514,
USA